$$
\begin{aligned}
r_i^n &= u_i^{n-k_i} \\
&\equiv \text{ a step input applied at } k_i\text{-th step} \qquad (43)
\end{aligned}
$$

where

$$
u_i^{n-k_i} \equiv \left\{ \begin{array}{ll} 0 & (n = 1, 2, \ldots, k_i - 1) \\ 1 & \qquad (n \geq k_i) \end{array} \right\} \qquad (44)
$$

To assure independence of the inputs, the onsets of the signals should be apart from each other by a sufficient number of steps, *i.e.*, k_i must be large enough.

Similarly, in another alternate approach, a pulse input based SCI (PSCI) may be used. In this approach, one can also apply multiple pulse inputs in a sequential manner:

$$
\begin{aligned}
r_i^n &= \delta_i^{n-k_i} \\
&\equiv \text{ a step input applied at } k_i\text{-th step} \qquad (45)
\end{aligned}
$$

where

$$
\delta_i^{n-k_i} \equiv \left\{ \begin{array}{ll} 1 & \qquad (n = k_i) \\ 0 & (n = \text{all other points}) \end{array} \right\} \qquad (46)
$$

Again, k_i should be large enough to ensure independency of the applied signals.

Yet another alternate embodiment to the SCI/ERA methods discussed above will now be described. In applications of discrete-time computational models, there exist two conflicting requirements for the incremental time step Δt. For numerical convergence, one should adopt a sufficiently small Δt_1. On the other hand, given the highest frequency of interest, ω_c, the Nyquist criterion requires $\Delta t_2 \leq \frac{\pi}{\omega_c}$. Usually, for practical purposes $\Delta t_2 >> \Delta t_1$.

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For instance, in a structural model that is coupled with a CFD model for aeroelastic applications, the highest mode usually has a much lower natural frequency than the highest frequency content in the aerodynamic model. If the signals used in the SCI/ERA methods are sharp as in the random, step, or pulse inputs, the SCI will excite all the system dynamics and hence this characteristic will be carried over to the ERA based reduced-order model. As a result, the ERA reduced-order model (ROM) may be prone to have a large dimension to span the same high frequency range as the original full-order model (FOM), which suggests that there may be room for further order reduction in the ROM.

To perform a second order reduction, one can apply the Frequency-Domain Karhunen-Loeve (FDKL) method to the ERA ROM defined by matrices, (28)-(31), wherein frequency samples of the system within the given frequency range, ($-\omega_c$, ω_c) are used to extract optimal modes, and a new reduced-order model is constructed via the Galerkin's approximation. (see, *e.g.*, Kim, T., *Discrete-Time Eigen Analysis and Optimal Model Reduction for Flutter & Aeroelastic Damping/Frequency Prediction Based On CFL3D-ERA*, B-ADVTECH-LLL-M02-013, BCAG, February 2003, incorporated herein by reference). In this embodiment, the optimal or so called KL modes $\O_i$, are the eigenmodes of the covariance matrix **K**:

$$\mathbf{K}\,\boldsymbol{\phi}_i = \lambda_i\,\boldsymbol{\phi}_i \tag{47}$$

where

$$\begin{aligned} K_{ij} &\equiv \boldsymbol{\mathcal{X}}(\omega_i)\,\boldsymbol{\mathcal{X}}(\omega_j)^{*T} && (48) \\ \omega_i &\equiv \text{sampling frequencies} \\ &= [\omega_1\ \omega_2\ \ldots\ \omega_M] && (49) \end{aligned}$$

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where $\omega_1 = -\omega_c$ and $\omega_M = \omega_c$. $X(\omega_i)$ are the frequency solutions of the ERA ROM subjected to the single-composite-input described by (20) and (21) except that it is prescribed in the frequency domain. Once the optimal modes are obtained, assume

$$\mathbf{x} \simeq \mathbf{\Phi}\,\mathbf{p} \tag{50}$$

where

$$\mathbf{\Phi} \equiv [\phi_1\ \phi_2\ \dots\ \phi_{R_1}] \tag{51}$$

$$\mathbf{p} \equiv \left\{ \begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_{R_1} \end{array} \right\} \tag{52}$$

R_1 is set to be equal to the rank of the covariance matrix which is usually smaller than R.

After inserting (50) into (1) and (2) with the ERA ROM matrices, premultiplying both sides by $\mathbf{\Phi}^T$ yields a new reduced-order model as

$$\mathbf{p}^{n+1} = \mathbf{A}_1\,\mathbf{p}^n + \mathbf{B}_1\,\mathbf{u}^n \tag{53}$$

$$\mathbf{y}^n = \mathbf{C}_1\,\mathbf{p}^n + \mathbf{D}\,\mathbf{u}^n \tag{54}$$

where

$$\mathbf{A}_1 \equiv \mathbf{\Phi}^T\,\mathbf{A}\,\mathbf{\Phi} \tag{55}$$

$$\mathbf{B}_1 \equiv \mathbf{\Phi}^T\,\mathbf{B} \tag{56}$$

$$\mathbf{C}_1 \equiv \mathbf{C}\,\mathbf{\Phi} \tag{57}$$

The dimension of the new model is $(R_1 \times R_1)$.

Application of various embodiments of methods and systems to representative, large-scaled CFD models will now be described. Unlike the general system described by Equations (1) and (2), an unsteady fluid dynamic system is driven by displacement and

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velocity of its moving boundary surface simultaneously as they both contribute to the normal downwash on the surface. If one considers a statically nonlinear, dynamically linearized flow field, the unsteady fluid motion can be described as

$$\mathbf{x}^{n+1} = \mathbf{A}\,\mathbf{x}^{n} + \mathbf{B}_0\,\mathbf{u}^{n} + \mathbf{B}_1\,\dot{\mathbf{u}}^{n} \tag{58}$$

$$\mathbf{y}^{n} = q\,(\mathbf{C}\,\mathbf{x}^{n} + \mathbf{D}_0\,\mathbf{u}^{n} + \mathbf{D}_1\,\dot{\mathbf{u}}^{n}) \tag{59}$$

where

$$\mathbf{x} \equiv (L \times 1) \text{ fluid states} \tag{60}$$

$$\mathbf{u} \equiv (N_i \times 1) \text{ generalized displacements} \tag{61}$$

$$\dot{\mathbf{u}} \equiv (N_i \times 1) \text{ generalized velocities} \tag{62}$$

$$\mathbf{y} \equiv (N_i \times 1) \text{ generalized aerodynamic forces} \tag{63}$$

$$q \equiv \text{dynamic pressure} \tag{64}$$

It is noted that the above equations progress in non-dimensional time, $\tau \equiv \frac{Vt}{b}$, rather than in the real time t and $(\dot{\ })$ is the first derivative with respect to τ. In fact, the dependency of the normal downwash on air speed disappears when the equations are discretized in τ, as in Equations (58) and (59). The structural degrees of freedom, u_i are the generalized coordinates associated with structural modes. These modes are used to describe the motion of the lifting surface. Two different types of reduced-order fluid dynamic models can be obtained depending on how the inputs are treated. If necessary, the FDKL/SCI can be performed for additional reduction.

In one particular embodiment, a method in accordance with the invention may be applied to an aerodynamic ROM with displacement and velocity inputs. In this embodiment, one can treat u^n and $\dot{u}^n$ separately as independent inputs. Thus, for the pulse inputs

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$$u_i^n = \dot{u}_i^n = \delta^n \;\equiv\; \left\{ \begin{array}{ll} 1 & (n=0) \\ 0 & (n=1,2,\ldots,M) \end{array} \right\} \tag{65}$$

for $i = 1,2,\ldots,N_i$, we obtain the corresponding responses $\mathbf{y}_i^0, \mathbf{y}_i^1$ at the first two time steps. Let us define

$$\mathbf{Y}^0 = [\mathbf{y}_1^0\, \mathbf{y}_2^0 \cdots \mathbf{y}_{N_i}^0\; \mathbf{y}_{N_i+1}^0\; \mathbf{y}_{N_i+2}^0 \cdots \mathbf{y}_{2N_i}^0] \tag{66}$$

$$\mathbf{Y}^1 = [\mathbf{y}_1^1\, \mathbf{y}_2^1 \cdots \mathbf{y}_{N_i}^1\; \mathbf{y}_{N_i+1}^1\; \mathbf{y}_{N_i+2}^1 \cdots \mathbf{y}_{2N_i}^1] \tag{67}$$

where the first N_i samples are due to the pulses in $\mathbf{u}^n$ and the next N_i ones are due to the pulses in $\dot{\mathbf{u}}^n$. Next, we prepare the following inputs,

$$\mathbf{b}_{SCI}^n \equiv \sum_{i=1}^{N_i} \mathbf{b}_{0\,i}\, r_i^n + \sum_{i=1}^{N_i} \mathbf{b}_{1\,i}\, r_{N_i+i}^n \tag{68}$$

$$\mathbf{d}_{SCI}^n \equiv \sum_{i=1}^{N_i} \mathbf{d}_{0\,i}\, r_i^n + \sum_{i=1}^{N_i} \mathbf{d}_{1\,i}\, r_{N_i+i}^n \tag{69}$$

Subject to the SCI, we sample the system response y^n and get

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$$
\begin{aligned}
\mathbf{y}_{c0}^{n} &\equiv \mathbf{y}^{n} - \sum_{i=1}^{2N_i} \mathbf{y}_i^0 \, r_i^n &\quad (70)\\
\mathbf{y}_{c1}^{n} &\equiv \mathbf{y}^{n+1} - \sum_{i=1}^{2N_i} \mathbf{y}_i^0 \, r_i^{n+1} - \sum_{i=1}^{2N_i} \mathbf{y}_i^1 \, r_i^n &\quad (71)
\end{aligned}
$$

Define

$$
\begin{aligned}
\mathbf{H}_{c0} &\equiv [\mathbf{y}_{c0}^1 \;\; \mathbf{y}_{c0}^2 \;\cdots\; \mathbf{y}_{c0}^{M-1}] &\quad (72)\\
\mathbf{H}_{c1} &\equiv [\mathbf{y}_{c1}^1 \;\; \mathbf{y}_{c1}^2 \;\cdots\; \mathbf{y}_{c1}^{M-1}] &\quad (73)
\end{aligned}
$$

where SVD of $\mathbf{H}_{c0}$ yields

$$
\begin{aligned}
\mathbf{H}_{c0} &\equiv \mathbf{U}\,\boldsymbol{\Sigma}\,\mathbf{V}^T \\
&\simeq [\mathbf{U}_R \;\; \mathbf{U}_D] \begin{bmatrix} \boldsymbol{\Sigma}_R & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_R^T \\ \mathbf{V}_D^T \end{bmatrix} \\
&= \mathbf{U}_R\,\boldsymbol{\Sigma}_R^{1/2}\,\boldsymbol{\Sigma}_R^{1/2}\,\mathbf{V}_R^T &\quad (74)
\end{aligned}
$$

with $R \equiv rank(\mathbf{H}_{c0})$. Hence, the reduced-order model is given by

$$
\begin{aligned}
\mathbf{D}_0 &= \text{the first } N_i \text{ columns of } \mathbf{Y}^0 &\quad (75)\\
\mathbf{D}_1 &= \text{the second } N_i \text{ columns of } \mathbf{Y}^0 &\quad (76)\\
\mathbf{C} &\simeq \mathbf{U}_R\,\boldsymbol{\Sigma}_R^{1/2} &\quad (77)\\
\mathbf{B}_0 &\simeq \text{the first } N_i \text{ columns of } \boldsymbol{\Sigma}_R^{-1/2}\,\mathbf{U}_R^T\,\mathbf{Y}^1 &\quad (78)\\
\mathbf{B}_1 &\simeq \text{the second } N_i \text{ columns of} \boldsymbol{\Sigma}_R^{-1/2}\mathbf{U}_R^T\mathbf{Y}^1 &\quad (79)\\
\mathbf{A} &\simeq \boldsymbol{\Sigma}_R^{-1/2}\,\mathbf{U}_R^T\,\mathbf{H}_{c1}\,\mathbf{V}_R\,\boldsymbol{\Sigma}_R^{-1/2} &\quad (80)
\end{aligned}
$$

In yet another particular embodiment, a method in accordance with the invention may be applied to an aerodynamic ROM with only the displacements as the system inputs. This is possible by applying simultaneously the pulse and double pulse inputs,

$$
u_i^n = \delta^n \equiv \left\{ \begin{array}{ll} 1 & (n = 0) \\ 0 & (n = 1, 2, \ldots, M) \end{array} \right\} \quad (81)
$$

$$
\dot{u}_i^n = \dot{\delta}^n \equiv \left\{ \begin{array}{ll} \frac{1}{\Delta\tau} & (n = 0) \\ -\frac{1}{\Delta\tau} & (n = 1) \\ 0 & (n = 2, 3, \ldots, M) \end{array} \right\} \quad (82)
$$

and get the corresponding responses $\mathbf{y}^0_{di}, \mathbf{y}^1_{di}$ at the first two time steps:

$$\mathbf{Y}^0_d = [\mathbf{y}^0_{d1} \;\; \mathbf{y}^0_{d2} \; \cdots \; \mathbf{y}^0_{dN_i}] \tag{83}$$

$$\mathbf{Y}^1_d = [\mathbf{y}^1_{d1} \;\; \mathbf{y}^1_{d2} \; \cdots \; \mathbf{y}^1_{dN_i}] \tag{84}$$

For a new SCI, we use

$$\mathbf{b}^n_{SCI} \equiv \sum_{i=1}^{N_i} \mathbf{b}_{0\,i}\, r^n_i + \sum_{i=1}^{N_i} \mathbf{b}_{1\,i}\, \dot{r}^n_i \tag{85}$$

$$\mathbf{d}^n_{SCI} \equiv \sum_{i=1}^{N_i} \mathbf{d}_{0\,i}\, r^n_i + \sum_{i=1}^{N_i} \mathbf{d}_{1\,i}\, \dot{r}^n_i \tag{86}$$

where $\dot{r}^n_i$ are discrete-time derivative of r^n_i. To be consistent with the use of the double pulse defined in (82), $\dot{r}^n_i$ are obtained by filtering r^n_i via $\dot{\delta}^n_i$, i.e.,

$$\dot{r}^n_i \equiv conv(r^k_i, \dot{\delta}^k_i) \tag{87}$$

which is equivalent to the backward difference equation,

$$\dot{r}^n_i \equiv \frac{r^n_i - r^{n-1}_i}{\Delta\tau} \tag{88}$$

Subject to the new SCI we sample the system response y^n and get

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